Lower and Upper Bounds of Cutoff Frequencies in Metallic Waveguides

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***Abstract*—It is shown how the upper and lower bounds of the cutoff frequencies of TE and TM modes in many waveguides bounded by metallic lossless walls and which may be hollow or comprise one or more inner conductors, can be computed using two independent methods. The methods are applicable whenever the cross-section of the waveguide can be split up into several regions bounded by lines having a fixed coordinate and includes several cases of practical interest. The theory is illustrated with reference to a rectangular coaxial line.**

I. Introduction

The MODE matching technique [1] occupies an important place among various numerical techniques used to solve microwave problems in general and waveguide problems in particular. It is applicable whenever the cross-section of the waveguide can be split up into several regions bounded by lines having a fixed coordinate in a separable system of coordinates.

Earlier [2] the author has presented some preliminary findings generalising his previous results [3]–[7] and facilitating the determination of the mode spectrum of a wide range of waveguides bounded by metallic walls. A few of the cross-sections, to which the theory to be discussed in what follows can be applied, are shown in Fig. 1.

It was noted earlier [2] that on the basis of preliminary results, it might be possible to deduce upper and lower bounds of the cutoff frequencies of various modes and this has now been confirmed. In addition, since the above paper [2] was written, the author realised that there was not one but two independent methods of arriving at the upper and lower bounds and this will be discussed in what follows.

The fact that upper and lower bounds of the lowest resonant TM mode of a re-entrant cylindrical cavity could be calculated using a mode matching procedure was noted by Taylor [8], whose findings were based on the work of Chu [9] but no attempt was made to generalize the results, nor has the matter been apparently followed up in published literature.

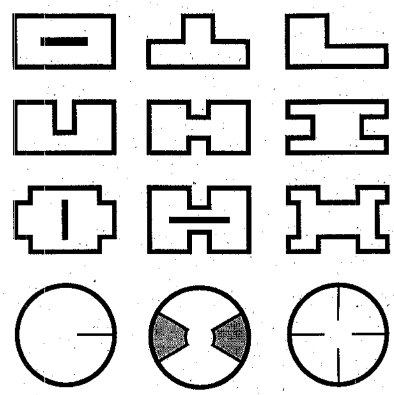

Fig. 1. A few waveguide cross-sections.

In what follows, it is proposed to deduce formulae facilitating the determination of the lower and upper bounds and illustrate the procedure with reference to a rectangular coaxial line, bounded by lossless metallic walls.

In the past the cutoff frequencies of rectangular coaxial lines have been determined by the author and other investigators [3], [4], [6], [10], [11]. In all cases only either the lower or the upper bound but not both were deduced.

The ability to determine both the upper and lower bounds, to be discussed in what follows, has the two-fold advantage of making it potentially possible to obtain a considerably more accurate result and reducing the time required to do so since only lower order determinants need to be considered.

II. General Theory

Consider a lossless uniform homogeneous waveguide having a cross-section which can be split up into two regions having fixed coordinates in an orthogonal system of coordinates. For the purpose of deducing the cutoff frequencies of TE modes, we express the longitudinal components of the magnetic field intensity H_z in the two respective regions (which need not be bounded by straight lines, as shown in Fig. 2) by the following two Fourier series:

$$H_{z1} = \sum_{l}^{\infty} A_l f_l(x_1)\phi_l^1(x_2) \tag{1}$$

$$H_{z2} = \sum_{m}^{\infty} B_m g_m(x_1)\phi_m^2(x_2). \tag{2}$$

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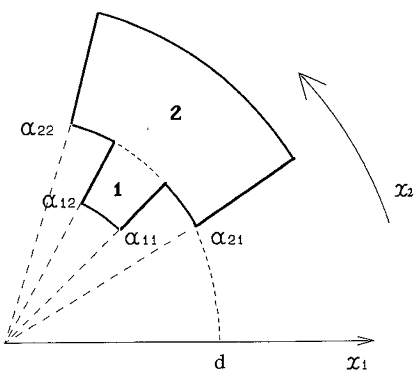

Fig. 2. A composite waveguide cross-section.

In the above formulae, x_1 and x_2 are coordinates of the mutually orthogonal axes of the system of coordinates (rectangular, cylindrical etc.)

The functions $\phi_l^1(x_1)$ and $\phi_m^2(x_2)$ satisfy orthogonality conditions of the form:

$$\langle w_1 \phi_m^1 \phi_n^1 \rangle \big|_{\alpha_{11}}^{\alpha_{12}} = K_1 \, \delta_{mn}$$

and

$$\langle w_2 \phi_m^2 \phi_n^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} = K_2 \, \delta_{mn}$$

in the two respective regions while w_1 and w_2 are appropriate weighting functions.

Moreover $f_l(x_1)$ and $g_m(x_1)$ do not fulfill any orthogonality conditions but are required to meet the boundary conditions of the particular problem while A_l and B_m are constant Fourier coefficients.

Let $x_1 = d$ represent the fixed coordinate of the boundary between the two regions in Fig. 2. In addition, with the view of simplifying the notation, we denote $f_{ld} = f_l(x_1 = d)$, $g_{md} = g_m(x_1 = d)$.

Furthermore f'_{ld} and g'_{md} represent the derivatives of f_l and g_m respectively, evaluated at $x_1 = d$.

Hence, using the above notation, the continuity of the longitudinal component of the magnetic field intensity at the common boundary is satsified, provided

$$\sum_l^{\infty} A_l f_{ld} \phi_l^1 = \sum_m^{\infty} B_m g_{md} \phi_m^2 \tag{3}$$

while the continuity of the transverse component of the electric field intensity implies that

$$\sum_l^{\infty} A_l f'_{ld} \phi_l^1 = \sum_m^{\infty} B_m g'_{md} \phi_m^2. \tag{4}$$

Remembering that a homogeneous lossless waveguide is being considered and both regions are filled with the same medium, multiplying (3) by $w_2 \phi_n^2$ and (4) by $w_1 \phi_k^1 (k = l)$ respectively and integrating shows that

$$\sum_l^{L} A_l f_{ld} \langle w_2 \phi_l^1 \phi_n^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \approx B_n g_{nd} \langle w_2 \phi_n^2 \phi_n^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \tag{5}$$

$$A_l f'_{ld} \langle w_1 \phi_l^1 \phi_l^1 \rangle \big|_{\alpha_{11}}^{\alpha_{12}} \approx \sum_m^{M} B_m g'_{md} \langle w_1 \phi_l^1 \phi_m^2 \rangle \big|_{\alpha_{11}}^{\alpha_{12}} = \sum_m^{M} B_m g'_{md} \langle w_1 \phi_l^1 \phi_m^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \tag{6}$$

when it is noted that along the boundaries α_{12} to α_{22} as well as α_{21} to α_{11} the electric field intensity vanishes. Moreover, for the purpose of arriving at a numerical solution, the infinite series in (3) and (4) must be truncated, and hence only L and M terms respectively are retained in (5) and (6).

When the coefficients A_l are eliminated from (5) and (6) and the order of summation is interchanged, we find that

$$E_{11} = \sum_m^M \sum_n^M B_m \left\{ g'_{md} \sum_l^L \frac{f_{ld} \langle w_1 \phi_l^1 \phi_m^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \langle w_2 \phi_l^1 \phi_n^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}}}{f'_{ld} \langle w_1 \phi_l^1 \phi_l^1 \rangle \big|_{\alpha_{11}}^{\alpha_{12}}} - \delta_{mn} g_{md} \langle w_2 \phi_m^2 \phi_m^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \right\} = 0. \tag{7}$$

Equations (7) have a solution only if their determinant vanishes and hence the eigenvalues (cutoff frequencies) are determined by equating the determinant of the $M \times M$ matrix to zero. It is clear that increasing the number of terms of the inner summation L is considerably less costly and time-consuming than increasing the order of the determinant M. Hence L can be made very much larger than M and the inner summation can be performed to a high degree of accuracy, especially if accelerating factors [4] are used.

With reference to (3) and (5), it is evident that the continuity of the magnetic field intensity at the common boundary surface can be almost perfectly satisfied. On the other hand the mismatch of the electric field intensity (6) can be expected to be much more severe if $M \ll L$.

Next, if we let

$$H_{z1} = \sum_m^M A_m f_m \phi_m^1 \tag{8}$$

$$H_{z2} = \sum_l^L B_l g_l \phi_l^2 \tag{9}$$

and hence

$$\sum_m^M A_m f_{md} \phi_m^1 \approx \sum_l^L B_l g_{ld} \phi_l^2 \tag{10}$$

$$\sum_m^M A_m f'_{md} \phi_m^1 \approx \sum_l^L B_l g'_{ld} \phi_l^2 \tag{11}$$

and multiply (10) by $w_2 \phi_k^2 (k = l)$ and (11) by $w_1 \phi_n^1$ and then perform the integration over the respective regions, then we find that

$$\sum_m^M A_m f_{md} \langle w_2 \phi_m^1 \phi_l^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \approx B_l g_{ld} \langle w_2 \phi_l^2 \phi_l^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}} \tag{12}$$

$$A_n f'_{nd} \langle w_1 \phi_n^1 \phi_n^1 \rangle \big|_{\alpha_{11}}^{\alpha_{12}} \approx \sum_l^L B_l g'_{ld} \langle w_1 \phi_n^1 \phi_l^2 \rangle \big|_{\alpha_{11}}^{\alpha_{12}} = \sum_l^L B_l g'_{ld} \langle w_1 \phi_n^1 \phi_l^2 \rangle \big|_{\alpha_{21}}^{\alpha_{22}}. \tag{13}$$

It will be noted that (8) and (9) differ from (1) and (2) only in that different indices are used; this has been done with the view of ensuring that the index l is reserved for the inner summation. Moreover the transformation on the right hand side of (13) follows from the fact that the electric field intensity vanishes along α_{12} to α_{22} and α_{21} to α_{11} when $x_1 = d$ (Fig. 2).

This time the Fourier coefficients B_l can be eliminated and we find that

$$E_{21} = \sum_{m}^{M}\sum_{n}^{M} A_m \left\{ f_{md} \sum_{l}^{L} \frac{g'_{ld}\langle w_1\phi_n^1\phi_l^2\rangle|_{\alpha 21}^{\alpha 22}\langle w_2\phi_m^1\phi_l^2\rangle|_{\alpha 21}^{\alpha 22}}{g_{ld}\langle w_2\phi_l^2\phi_l^2\rangle|_{\alpha 21}^{\alpha 22}} - \delta_{mn} f'_{md}\langle w_1\phi_m^1\phi_m^1\rangle|_{\alpha 11}^{\alpha 12} \right\} = 0. \quad (14)$$

Noting again that L can be made very much larger than M, with reference to (11) and (13) we observe that this time the continuity of the electric field intensity along the common boundary can be almost perfectly satisfied, while the magnetic field intensity on both sides of the boundary is much more severely mismatched.

When equations E_{11} and E_{21} are used to arrive at numerical results, it becomes apparent that as the size of the determinant (and hence M) is increased, the magnitude of the cutoff frequency increases for E_{11} and decreases for E_{21}; in other words, while E_{11} makes it possible to deduce the lower bound of the cutoff frequency, an upper bound is obtained with the aid of E_{21}.

This result may be understood with reference to the work of Chu [9] and was noted in one specific instance by Taylor [8].

Instead of proceeding as above, one can multiply equation (3) by $w_1\phi_k^1$ (letting $k = l$), (4) by $w_2\phi_n^2$ to be followed up in both cases by the appropriate integrations.

Proceeding in a similar manner as above, we find that

$$E_{12} = \sum_{m}^{M}\sum_{n}^{M} B_m \left\{ g_{md} \sum_{l}^{L} \frac{f'_{ld}\langle w_2\phi_l^1\phi_n^2\rangle|_{\alpha 21}^{\alpha 22}\langle w_1\phi_l^1\phi_m^2\rangle|_{\alpha 11}^{\alpha 12}}{f_{ld}\langle w_1\phi_l^1\phi_l^1\rangle|_{\alpha 11}^{\alpha 12}} - \delta_{mn} g'_{md}\langle w_2\phi_m^2\phi_m^2\rangle|_{\alpha 21}^{\alpha 22} \right\} = 0. \quad (15)$$

Furthermore, when (10) and (11) are multiplied by $w_1\phi_n^1$ as well as $w_2\phi_k^2$ respectively (letting $k = l$) and the appropriate integrations are performed, we can eliminate the B_l Fourier coefficients instead to deduce that

$$E_{22} = \sum_{m}^{M}\sum_{n}^{M} A_m \left\{ f'_{md} \sum_{l}^{L} \frac{g_{ld}\langle w_1\phi_n^1\phi_l^2\rangle|_{\alpha 11}^{\alpha 12}\langle w_2\phi_m^1\phi_l^2\rangle|_{\alpha 11}^{\alpha 22}}{g'_{ld}\langle w_2\phi_l^2\phi_l^2\rangle|_{\alpha 21}^{\alpha 22}} - \delta_{mn} f_{md}\langle w_1\phi_m^1\phi_m^1\rangle|_{\alpha 11}^{\alpha 12} \right\} = 0. \quad (16)$$

It will be noted that (7) and (15) as well as (14) and (16) respectively have a similar but not identical form. One major difference are different limits of the two inner products of the inner sum evident in E_{12} and E_{22}. The two inner products of E_{12} and E_{22} have the same limits only if $\alpha_{11} = \alpha_{21}$ and $\alpha_{12} = \alpha_{22}$, while no such considerations apply to E_{11} and E_{21}. As an example, when a rectangular coaxial line is considered, only a single inner product must be evaluated using (9) and (14) irrespective of whether the inner conductor is infinitesimally thin, or not. On the other hand, using (15) and (16) two different inner products are required, unless the conductor is infinitesimally thin, in which case one is sufficient.

When eigenvalues (cutoff frequencies) are calculated with the aid of E_{12} and E_{22}, we find that as the size of the determinant (i.e., M) is increased, the eigenvalues decrease when E_{12} is used and increase when E_{22} is applied; the reasons are similar to those discussed with reference to E_{11} and E_{21}.

To put it differently, E_{12} yields an upper bound while E_{22} yields a lower bound.

Hence there are effectively two independent methods of arriving at the lower and upper bounds of an eigenvalue, using either E_{11} or E_{22} and E_{21} or E_{12}, respectively.

Next we consider the TM modes. In this case (1) and (2) are replaced by

$$E_{z1} = \sum_{l}^{\infty} A_l f_l(x_1)\phi_l^1(x_2) \quad (17)$$

$$E_{z2} = \sum_{m}^{\infty} B_m g_m(x_1)\phi_m^2(x_2) \quad (18)$$

and (8) and (9) must be modified in a similar manner.

Evidently functions f_l, g_m, ϕ_l^1 and ϕ_m^2, which must satisfy different boundary conditions, as well as the Fourier coefficients A_l and B_m are no longer the same as in (1) and (2) or (8) and (9) for that matter. However the derivation of the results follows in an analogous manner and expressions (7) for E_{11}, (14) for E_{21}, (15) for E_{12} and (16) for E_{22} still hold, noting that all symbols must be given a different interpretation. Moreover, it is clear that whenever the continuity of the electric field intensity is satisfied almost perfectly for TE modes, the continuity of the magnetic field intensity is satisfied by TM modes and vice versa. Designating the left hand side of (7), (14), (15), and (16), respectively as M_{11}, M_{21}, M_{12} and M_{22} for TM modes, we find for example that while E_{11} yields a lower bound, the cutoff frequency calculated with the aid of M_{11} (having the same form as E_{11} apart from the interpretation of symbols) will yield the upper bound etc.

All the above results were derived assuming the presence of a single common boundary between two regions (Fig. 2). Considerations of symmetry make it possible to extend the range of waveguide cross-sections to which the above theory can be applied and a rectangular coaxial line having a symmetrically located inner conductor [3] may serve as an example.

Finally the above theory can be readily extended to situations when it is necessary to consider three regions and hence there are two boundaries along which the field must be matched. In this case two equations from the above set (7), (14), (15), and (16) must be solved simultaneously and crossed rectangular coaxial structures [5] may serve as an example.

III. Rectangular Coaxial Lines

As an example, to illustrate the above theory, we consider a symmetric rectangular coaxial line for which cutoff frequencies are well known [3]–[4], [6], [10]–[12].

Using the notation of the author's earlier paper [3], we note that (Fig. 3) so long as the inner conductor is symmetrically located with respect to the outer conductor, it

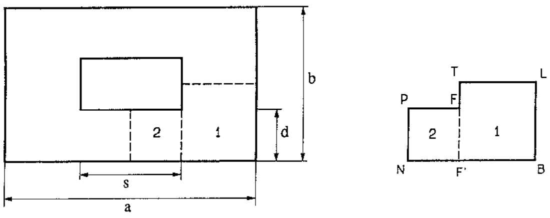

Fig. 3. Cross-section of a rectangular coaxial waveguide.

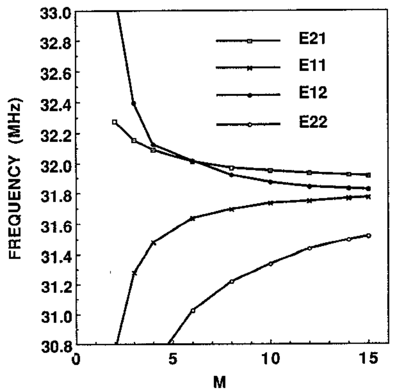

Fig. 4. TE_{11} mode cutoff frequency of a rectangular coaxial waveguide (a = 6 m, b = 6 m, s = 5 m, inner conductor infinitesimally thin) as a function of the determinant size M.

is possible to consider only a quarter of the cross-section (Fig. 3(b)), and thereby simplify the analysis. In this case the field must be matched along FF′, while PN and TL are electric or magnetic walls, depending on mode order. Moreover, when the inner conductor is infinitesimally thin, then it may be considered as an electric wall and some modes have cutoff wavelengths which are the same [4] as those of a hollow waveguide having the same dimensions a and b.

For comparison purposes, a cross-section discussed by de Leo *et al.* [10] having a width a = 6 m, a height b = 6 m (using symbols of Fig. 3, which may differ from those of [10]) and an infinitesimally thin inner conductor having a width s = 5 m, was investigated.

Fig. 4 shows the dependence of the cutoff frequency of the TE_{11} mode in the above structure on the size of the determinant, using all four expressions (7), (14), (15), and (16).

According to [10], the cutoff frequency deduced by the authors was 31.82 MHz, while the frequency reported by Tippet *et al.* was 32.4 MHz and that by Mittra *et al.* 32.0 MHz.

Similar computations were performed for other modes, including the TE_{01} and TE_{21} modes and results quoted in [10] were found to fall within the range of the lower and upper bounds. Computations were also carried out for the TE_{10} mode of a rectangular coaxial line with an inner conductor of finite width and results compared with those deduced by Pyle [2], [3] who used finite differences; Pyle derived his result for a single ridge waveguide but it is clear that the cutoff wavelength of the TE_{10} mode is the same as that of the appropriate rectangular coaxial line.

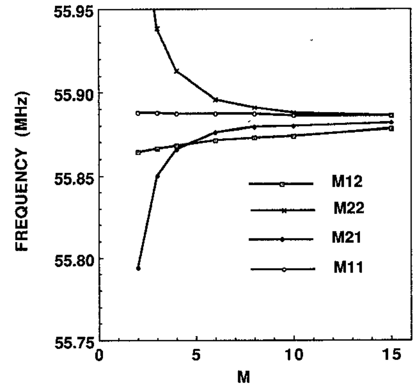

Fig. 5. TM_{11} mode cutoff frequency of a rectangular coaxial waveguide (a = 6 m, b = 6 m, s = 5 m, inner conductor infinitesimally thin) as a function of the determinant size M.

The effect of varying the size of the determinant on the cutoff frequency of the TM_{11} mode in the structure for which the cutoff frequency of the TE_{11} mode was investigated is shown in Fig. 5; again, expressions (7), (14), (15), and (16) were used for that purpose but no comparison figures are available in [10].

All numerical results were obtained using the EPFL VAX computer using single precision arithmetic.

The rate of convergence of the inner summation was found to vary considerably, depending on which of the four formulae (7), (14), (15), or (16) was used and on the mode being investigated; if necessary, convergence can be improved using accelerating factors [4]. In all cases the number of terms of the inner summation was increased until it became apparent that a further increase would have no effect upon the result.

Moreover, the above findings suggest that the rate of convergence of the end result (cutoff frequency or wavelength) again varies considerably, depending on which expression is used in any particular case.

Under no circumstances were the curves obtained using inner and upper bound formulae ever found to cross. Thus, for example, referring to Fig. 4 it can be confidently expected that the cutoff frequency of the TE_{11} mode lies between 31.797 and 31.817 MHz. If desired, the range of uncertainty could have been further reduced but the immediate objective of this paper is to demonstrate the technique rather than obtain a highly accurate result for a particular mode and some particular geometry.

IV. Conclusion

It has been shown that the lower and upper bounds of the cutoff frequencies of the TE and TM modes of a large number of different waveguides can be readily deduced using two different methods in each instance.

While the rate of convergence depends on a particular mode and geometry, the technique has the potential of yielding highly precise results limited only by such factors as computer word length and roundoff errors.

The technique is applicable to a large number of different geometries and cross-sections and it is strongly believed that it has the potential to be used as a benchmark, facilitating the assessment and comparison between other generally applicable methods for which lower and upper bounds are not readily available or are not adequate.

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